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Slow nonlinear oscillations in a circular well

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The period T of the Helmholtz mode in a circular well that is bounded above by a free surface and below by a semi-infinite reservoir is determined in terms of elliptic integrals. It is shown that T decreases monotonically with increasing amplitude A and is within 1% (10%) of the linear limit T_0 for $A/h_0 < 0.4(1.0)$, where h_0 is the ambient depth.

1. Introduction

Informed by the work of Molin (2001), Miles (2002*a*, *b*), and Hirata & Craik (2003), I consider here the Helmholtz mode ($\omega^2 \ll g/a$) in a circular well of radius a and ambient depth h_0 that is bounded above by a free surface and below by a semi-infinite reservoir. This mode is source-like, and the mass flux in the well is balanced by an asymptotic inflow in the reservoir.

My primary aim is the determination of the period T of free oscillations of amplitude A for $0 < A < h_0$. After constructing the velocity potentials in the well and the reservoir in §§2 and 3, respectively, I invoke conservation of energy in §4 to obtain $h(t)$ and T in terms of elliptic integrals and show that T decreases monotonically with increasing A and is within 1% (10)% of the linear limit T_0 for $A/h_0 < 0.4(1.0)$.

2. Fluid motion in the well

The fluid in the well is bounded by $r = a$ and $0 < z < h(t)$, where r and z are cylindrical coordinates, and h is the elevation of the free surface above the mouth ($z = 0$). Neglecting the lateral variation of the motion (which therefore is rigid-body like), we obtain the velocity potential

$$\phi = \phi_0 + \dot{h}z \quad (0 \leqslant z \leqslant h), \tag{2.1}$$

where ϕ_0 is the potential at $z = 0+$, and $\dot{h} \equiv \mathrm{d}h/\mathrm{d}t$. Anticipating

$$\phi_0 = \alpha a \dot{h}, \quad \alpha \equiv \frac{8}{3\pi}, \tag{2.2a, b}$$

which follows from the matching of (2.1) with the solution in the reservoir (see §3), we obtain

$$\phi = \dot{h}(z + \alpha a) \quad (0 < z \leqslant h). \tag{2.3}$$

The perturbation pressure p, which is linear in z and must vanish at the free surface, is given by

$$p = \rho(g + \ddot{h})(h - z) \quad (0 \leqslant z \leqslant h). \tag{2.4}$$

3. Fluid motion in the reservoir

The velocity potential in the reservoir is determined by

$$\nabla^2\phi = 0 \quad (-\infty < z < 0), \tag{3.1}$$

$$\phi_z = \begin{cases} \dot{h} \\ 0 \end{cases} \quad (r \lessgtr a, z = 0-), \tag{3.2a}$$

and

$$\phi \to 0 \quad (R \equiv (r^2 + z^2)^{1/2} \to \infty, z < 0). \tag{3.2b}$$

Invoking the Green's function

$$G(\boldsymbol{r}, \boldsymbol{\rho}) = \frac{1}{2\pi|\boldsymbol{r} - \boldsymbol{\rho}|}, \tag{3.3}$$

we obtain (cf. Rayleigh 1896, § 302)†

$$\phi(\boldsymbol{r}) = \frac{1}{2\pi}\iint \frac{\boldsymbol{n}\cdot\nabla\phi(\boldsymbol{\rho})\,\mathrm{d}S(\boldsymbol{\rho})}{|\boldsymbol{r} - \boldsymbol{\rho}|} = \frac{\dot{h}}{2\pi}\iint \frac{\mathrm{d}S(\boldsymbol{\rho})}{|\boldsymbol{r} - \boldsymbol{\rho}|} \equiv \dot{h}\Phi(\boldsymbol{r}), \tag{3.4}$$

where $\boldsymbol{r}$ specifies the point of observation, $\boldsymbol{\rho}$ specifies a point on the boundary of the fluid, $\boldsymbol{n}$ is the outwardly directed normal, and the integral is over the boundary. The volumetric flux $\pi a^2\dot{h}$ through the mouth is balanced by the radial inflow $-2\pi R^2\phi_R$, which may be calculated from the asymptotic approximation

$$\phi \sim \frac{\dot{h}}{2\pi}\left[\frac{\pi a^2}{R}\right] = \tfrac{1}{2}\dot{h}\left[\frac{a^2}{R}\right] \quad (R \to \infty, z < 0). \tag{3.5}$$

This inflow may be associated with some outer motion, as in Molin's (2001) treatment of a rectangular moonpool.

The neglect of the radial variation of the motion in the well prevents an exact matching with the motion in the reservoir, but closure may be effected within the present approximation by matching the total impulse (cf. Lamb 1932, § 196) or, equivalently, the average of ϕ, in the mouth:

$$\langle\phi\rangle = \begin{cases} \phi_0 \\ \langle\Phi(\boldsymbol{r})\rangle\dot{h} \end{cases} \quad (0 < r < a, z = 0\pm), \tag{3.6}$$

where $\langle\,\rangle$ signifies an average over the mouth. Invoking

$$\langle\Phi(\boldsymbol{r})\rangle = \alpha a, \quad \alpha \equiv \frac{8}{3\pi} \tag{3.7}$$

(Rayleigh 1896, §§ 302, 312), we obtain (2.2).

The pressure in the reservoir is determined by the Bernoulli equation

$$\frac{p}{\rho} \equiv gh_0 - \left[\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz\right] \tag{3.8a}$$

$$= g(h_0 - z) - \ddot{h}\Phi - \tfrac{1}{2}\dot{h}^2(\nabla\Phi)^2. \tag{3.8b}$$

4. Energy integrals

The energies of the fluid motion in the well and the reservoir are given by

$$E_w = \tfrac{1}{2}m[g(h - h_0)^2 + h\dot{h}^2], \tag{4.1}$$

† Rayleigh considers low-frequency acoustical radiation from a flanged, open pipe, which is analogous to the present problem.

and

$$E_r = \tfrac{1}{2}\rho \iint_S \phi(\boldsymbol{n}\cdot\nabla\phi)\,\mathrm{d}S = \tfrac{1}{2}m\alpha\, a\dot{h}^2, \tag{4.2}$$

where

$$m \equiv \pi a^2 \rho. \tag{4.3}$$

Requiring the total energy to be conserved, we obtain

$$E_w + E_r = \tfrac{1}{2}m[g(h-h_0)^2 + (h+\alpha a)\dot{h}^2] \tag{4.4a}$$
$$= \tfrac{1}{2}mgA^2, \tag{4.4b}$$

where A is the amplitude of the oscillation of the free surface about $z = h_0$.

Solving (4.4) for

$$\frac{\mathrm{d}t}{\mathrm{d}h} = \pm g^{-1/2}\left[\frac{h+\alpha a}{A^2-(h-h_0)^2}\right]^{1/2}, \quad (0 < h_0 - A < h < h_0 + A), \tag{4.5}$$

integrating, choosing $t = 0$ at $h = h_0$, and letting

$$h = h_0 + A\sin\theta \quad \left(-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi\right), \tag{4.6}$$

we obtain the elliptic integral (Byrd & Friedman 1954, 288.01)

$$t = g^{-1/2}\int_0^{\theta}(h_1 + A\sin\psi)^{1/2}\,\mathrm{d}\psi, \tag{4.7}$$

where

$$h_1 \equiv h_0 + \alpha a. \tag{4.8}$$

The period is given by

$$T = 2g^{-1/2}\int_{-\pi/2}^{\pi/2}(h_1 + A\sin\psi)^{1/2}\,\mathrm{d}\psi \tag{4.9a}$$
$$= 4\left[\frac{h_1+A}{g}\right]^{1/2} E(k), \quad k = \left[\frac{2A}{h_1+A}\right]^{1/2}, \tag{4.9b, c}$$

where E is a complete elliptic integral of the second kind and modulus k. Introducing

$$T_0 = 2\pi(h_1/g)^{1/2}, \tag{4.10}$$

we find that T/T_0 decreases monotonically from 1 for $A/h_1 \to 0$ through 0.983 for $A/h_1 = 1/2$ to $2\sqrt{2}/\pi = 0.900$ for $A/h_1 \uparrow 1$ (although $A < h_0 < h_1$ for the physical problem).

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